Local systems over complements of hyperplanes and the Kac–Kazhdan conditions for singular vectors

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In this paper we strenghten a theorem by Esnault-Schechtman-Viehweg, [3], which states that one can compute the cohomology of a complement of hyperplanes in a complex affine space with coefficients in a local system using only logarithmic global differential forms, provided certain "Aomoto non-resonance conditions" for monodromies are fulfilled at some "edges" (intersections of hyperplanes). We prove that it is enough to check these conditions on a smaller subset of edges, see Theorem 4.1.

We show that for certain known one dimensional local systems over configuration spaces of points in a projective line defined by a root system and a finite set of affine weights (these local systems arise in the geometric study of Knizhnik-Zamolodchikov differential equations, cf. [8]), the Aomoto resonance conditions at non-diagonal edges coincide with Kac-Kazhdan conditions of reducibility of Verma modules over affine Lie algebras, see Theorem 7.1.

1 Quasiisomorphism.

Let $\{H_i\}_{i\in I}$ be an affine arrangement of hyperplanes, i.e., $\{H_i\}_{i\in I}$ is a finite collection of (distinct) hyperplanes in the affine complex space $\mathbb{C}^n$. Define $U = \mathbb{C}^n - \bigcup_{i\in I} H_i$. We denote by Ω^p_U the sheaves of holomorphic forms on U for $0 \le p \le n$. We set $\mathcal{O}_U := \Omega^0_U$.

For any $i \in I$, choose a degree one polynomial function f_i on $\mathbb{C}^n$ whose zero locus is equal to H_i. Define $\omega_i := d\log f_i = df_i/f_i \in \Gamma(U, \Omega^1_U)$. For a given $r \in \mathbb{N} - \{0\}$ we choose matrices $P_i \in \operatorname{End}\mathbb{C}^s, i \in I$. Define

$$\omega := \sum_{i\in I} \omega_i \otimes P_i \in \Gamma(U, \Omega^1_U) \otimes \operatorname{End}\mathbb{C}^s.$$

The form ω defines the connection $d+\omega$ on the trivial bundle $\mathcal{E} := \mathcal{O}^s_U$. We suppose that $(d+\omega)$ is *integrable* which is equivalent to the condition $\omega\wedge\omega = 0$ as $d\omega = 0$. Let $\Omega^\bullet_U(\mathcal{E}) = \Omega^\bullet_U \otimes_{\mathcal{O}_U} \mathcal{E}$ be the de Rham complex with the differential $d+\omega$.

Define finite dimensional subspaces

$$A^p \subset \Gamma(U, \Omega^p_U(\mathcal{E})) = \Gamma(U, \Omega^p_U) \otimes_{\mathbf{C}} \mathbb{C}^s$$

as the $\mathbb{C}$-linear subspaces generated by all forms $\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}\otimes v$, $v \in \mathbb{C}^s$. Then the exterior product by ω defines

$$A^\bullet : 0 \longrightarrow A^0 \xrightarrow{\omega} A^1 \xrightarrow{\omega} \cdots \xrightarrow{\omega} A^n \longrightarrow 0$$

as a subcomplex of $\Gamma(U, \Omega_U^\bullet(\mathcal{E}))$.

Let $\overline{\mathbb{C}}^n$ be any smooth compactification of $\mathbb{C}^n$ such that H_∞ is a divisor. Write $H = H_\infty \cup (\bigcup_{\in I} H_i)$. Then $U = \overline{\mathbb{C}}^n - H$. (Typical examples for $\overline{\mathbb{C}}^n$ include the complex projective space $\mathbf{P}^n$, $(\mathbf{P}^1)^n$ and any toric manifold.) Note that $\omega \in \Gamma(U, \Omega_U^1) \otimes \operatorname{End} \mathbb{C}^s$ can be uniquely extended to be an $\operatorname{End} \mathbb{C}^s$- coefficient rational 1-form $\overline{\omega}$ on $\mathbb{C}^n$.

Theorem 1.1 *Suppose $\pi : X \to \overline{\mathbb{C}}^n$ be a blowing up of $\overline{\mathbb{C}}^n$ with centers in H such that 1) X is nonsingular, 2) $\pi^{-1}H$ is a normal crossing divisor, and 3) none of the eigenvalues of the resideu of $\pi^{-1}\bar{\omega}$ along any component of $\pi^{-1}H$ lies in $\mathbb{N} - \{0\}$. Then the inclusion*

$$A^\bullet \hookrightarrow \Gamma(U, \Omega_U^\bullet(\mathcal{E}))$$

is quasiisomorphism.

Proof. Same as the proof of the first theorem in [3]. □

2 Decomposable arrangements

Let $\mathcal{A}$ be a central arrangement in V, i.e., a finite collection of hyperplanes with $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$. Then $\mathcal{A}$ is called **decomposable** if there exist nonempty subarrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ (disjoint) and, after a certain linear coordinate change, defining equations for $\mathcal{A}_1$ and $\mathcal{A}_2$ have no common variables.

Let $\mathcal{A}$ be a nonempty central arrangement in $\mathbb{C}^n$. Let $T = \bigcap_{A \in \mathcal{A}} A \neq \emptyset$. Suppose $\operatorname{codim} T = k+1 > 0$. Then the points of $\mathbf{P}_T := \mathbf{P}^k$ parametrize the $(\dim X + 1)$-dimensional linear subspaces of $\mathbb{C}^n$ which contain T. In particular, if H is a hyperplane containing T, it uniquely determines a hyperplane H' in $\mathbf{P}^k$. Define $P(\mathcal{A}) := \mathbf{P}^k - \bigcup_{H \in \mathcal{A}} H'$.

Definition 2.1 *Define the* **beta invariant** *of a central arrangement $\mathcal{A}$ by*

$$\beta(\mathcal{A}) = (-1)^r \chi(P(\mathcal{A}))$$

where χ denotes the Euler characteristic.

Let $L(\mathcal{A})$ be the set of all edges of $\mathcal{A}$. We regard $L(\mathcal{A})$ as a lattice with the reverse inclusion as its partial order. Then $\mathbb{C}^n$ itself is the minimum element of $L(\mathcal{A})$. Let μ be the Möbius function of $L(\mathcal{A})$.

Definition 2.2 *([7, Def.2.52]) Define the characteristic polynomial of $\mathcal{A}$ by*

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(V, X) t^{\dim X}.$$

Proposition 2.3

$$\beta(\mathcal{A}) = (-1)^k \frac{d}{dt}\chi(\mathcal{A}, 1).$$

Proof. Since $P(\mathcal{A})$ is homotopy equivalent to the complement of of the decone $d\mathcal{A}$ [7, p.15] of $\mathcal{A}$ by [7, Prop. 2.51, Thm.5.93], one has

$$(1+t)\mathrm{Poin}(P(\mathcal{A}), t) = \mathrm{Poin}(U, t),$$

where U is the complement of $\mathcal{A}$ and Poin stands for the Poincaré polynomial. Thus, by [7, Def. 2.52],

$$\begin{aligned}(t-1)^{-1}\chi(\mathcal{A}, t) &= (t-1)^{-1}t^{\ell}\mathrm{Poin}(U, -t^{-1}) \\ &= (t-1)^{-1}t^{\ell}(1-t^{-1})\mathrm{Poin}(P(\mathcal{A}), -t^{-1}) \\ &= t^{\ell-1}\mathrm{Poin}(P(\mathcal{A}), -t^{-1}).\end{aligned}$$

Take the limit as t approaches 1. (Note $\chi(\mathcal{A}, 1) = 0$.) $\square$

Proposition 2.3 shows that the beta invariant for the matroid determined by $\mathcal{A}$. The beta invariant for a matroid was introduced by Crapo [2].

Theorem 2.4 *([2, Theorem 2])*
(1) If $\mathcal{A}$ is not empty, then $\beta(\mathcal{A}) \geq 0$.
(2) $\beta(\mathcal{A}) = 0$ if and only if $\mathcal{A}$ is decomposable. $\square$

Let $\mathcal{A}$ be an affine arrangement of hyperplanes in $\mathbb{C}^n$. Let L be an edge of $\mathcal{A}$.

Definition 2.5 *An edge L is called* **dense** *in $\mathcal{A}$ if and only if the central arrangement*

$$\mathcal{A}_L := \{A \in \mathcal{A} \mid L \subseteq A\}$$

is not decomposable.

By Theorem 2.4, we have

Proposition 2.6 *Let $L \in L(\mathcal{A})$ with* $\mathrm{codim}L = r+1$. *Then the following conditions are equivalent:*
(1) L is dense,
(2) $\mathcal{A}_L$ is not decomposable,
(3) $\chi(P(\mathcal{A}_L)) \neq 0$,
(4) $\beta(\mathcal{A}_L) := (-1)^r\chi(P(\mathcal{A}_L)) > 0$. $\square$

3 Resolution of a hyperplanelike divisor

Let Y be a smooth complex compact manifold of dimension n, $\mathcal{D}$ a divisor. $\mathcal{D}$ is **hyperplanelike** if Y can be covered by coordinate charts such that in each chart $\mathcal{D}$ is a union of hyperplanes. Such charts will be called **linearizing**.

Let $\mathcal{D}$ be a hyperplanelike divisor, U a linearizing chart. A **local edge** of $\mathcal{D}$ in U is any nonempty irreducible intersection in U of hyperplanes of $\mathcal{D}$ in U. An **edge** of $\mathcal{D}$ is the maximal analytic continuation in Y of a local edge. Any edge is an immersed submanifold in Y. An edge is called **dense** if it is locally dense.

For $0 \leq j \leq n-2$, let $\mathcal{L}_j$ be the collection of all dense edges of $\mathcal{D}$ of dimension j. The following theorem is essentially in [10, 10.8].

Theorem 3.1 *Let $W_0 = Y$. Let $\pi_1 : W_1 \to W_0$ be the blow up along points in $\mathcal{L}_0$. In general, for $1 \leq s \leq \ell - 1$, let $\pi_s : W_s \to W_{s-1}$ be the blow up along the proper transforms of the $(s-1)$-dimensional dense edges in $\mathcal{L}_{s-1}$ under $\pi_1 \circ \cdots \circ \pi_{s-1}$. Let $\pi = \pi_1 \circ \cdots \circ \pi_{n-1}$. Then $W := W_{n-1}$ is nonsingular and $\pi^{-1}(\mathcal{D})$ normal crossing.*

4 Arrangements in $\mathbf{P}^n$

Let $\{H_i\}_{i \in I}$ be an affine arrangement of hyperplanes in $\mathbb{C}^n$. Recall $U, f_i, \omega_i, P_i, \omega, \mathcal{E}$, and $A^\bullet$ from Section 1. Choose $\mathbf{P}^n$ as the compactification of $\mathbb{C}^n$. Let $H_\infty = \mathbf{P}^n - \mathbb{C}^n$ and $\mathcal{A} = \{\overline{H}_i\}_{i \in I} \cup \{H_\infty\}$. ($\overline{H}_i$ is the closure of H_i in $\mathbf{P}^n$.) Obviously $\left(\bigcup_{i \in I} \overline{H}_i\right) \cup H_\infty$ is a hyperplanelike divisor. Suppose $(z_0 : \cdots : z_n)$ be a homogeneous coordinate system with $H_\infty : z_0 = 0$. Then each ω_i is uniquely extended to be a rational form $\overline{\omega}_i$ on $\mathbf{P}^n$; $\overline{\omega}_i = \omega_i - (dz_0/z_0)$. Thus the form $\omega = \sum_{i \in I} \omega_i \otimes P_i \in \Gamma(U, \Omega^1_U) \otimes \operatorname{End} \mathbb{C}^s$. can be uniquely extended to $\overline{\omega}$:

$$\overline{\omega} = \sum_{i \in I} \overline{\omega}_i \otimes P_i = \sum_{i \in I} \omega_i \otimes P_i - (dz_0/z_0) \otimes \left(\sum_{i \in I} P_i \right).$$

Define $P_\infty = -\sum_{i \in I} P_i$. For any edge L of $\mathcal{A}$, let $I_L = \{i \in I \cup \{\infty\} | L \subseteq H_i\}$. Let $P_L := \sum_{i \in I_L} P_i$. By Theorems 1.1 and 3.1, we get

Theorem 4.1 *We set $\mathcal{L}$ be the set of all dense edges of $\mathcal{A}$. Suppose that*
(Mon)* : *for all $L \in \mathcal{L}$, none of the eigenvalues of P_L lies in $\mathbb{N} - \{0\}$.*
Then the inclusion

$$A^\bullet \hookrightarrow \Gamma(U, \Omega^\bullet_U(\mathcal{E}))$$

is quasiisomorphism. $\square$

Remark. Since "dense" implies "bad" [3], Theorem 4.1 improves the main theorem of [3].

Corollary 4.2 *Under the assumption of Theorem 4.1, one has*

$$H^p(U, \mathcal{S}) \cong H^p(A^\bullet) \quad \text{for} \quad 0 \leq p \leq n$$

where $\mathcal{S}$ is the local system of flat sections of $(\mathcal{E}, d + \omega)$ on U. $\square$

Corollary 4.3 *Suppose that*

(Mon) :** *for all $L \in \mathcal{L}$, none of the eigenvalues of P_L lies in $\mathbb{N} \cup \{0\}$.*

Also suppose that $P_iP_j = P_jP_i$ for all i, j. Then

$$H^p(U, \mathcal{S}) = 0 \quad \text{for} \;\; p \neq n.$$

Proof. By Theorem 4.1 and [11, 4.1]. □

5 Discriminantal arrangements in $(\mathbf{P}^1)^n$

See [8] for discriminantal arrangments.

Let Γ be a graph without loops with vertices $v_1, \ldots, v_p$. Let $n_1, \ldots, n_r$ be nonnegative integers, $n = n_1 + \cdots + n_r$, $X = \{(i, \ell) | \ell = 1, \ldots, r, i = 1, \ldots, n_\ell\}$, $Y = (\mathbf{P}^1)^n$. Label the factors of Y by elements of X and for every $(i, \ell) \in X$ fix an affine coordinate $t_i(\ell)$ on the (i, ℓ)-th factor.

For pairwise distinct $z_1, \ldots, z_k \in \mathbb{C}$, $z_{k+1} = \infty$, introduce in Y a **discriminantal arrangement** $\mathcal{A}$ of "hyperplanes"

$$H_{(i,\ell),j} : t_i(\ell) = z_j \text{ for } (i, \ell) \in X, j = 1, \ldots, k + 1,$$

$$H_{(i,\ell),(j,\ell)} : t_i(\ell) = t_j(\ell) \text{ for } 1 \leq i < j \leq n_\ell,$$

and

$$H_{(i,\ell),(j,m)} : t_i(\ell) = t_j(m)$$

for ℓ, m such that v_ℓ and v_m are joined by an edge in the graph and $i = 1, \cdots, n_\ell$, $j = 1, \cdots, n_m$. The union of these "hyperplanes" is a hyperplanelike divisor. Let $\Delta \subseteq \Gamma$ be a connected subgraph with vertices labelled by $V \subseteq \{1, \ldots, r\}$. For every $\ell \in V$ fix a nonempty subset $I_\ell \subseteq \{1, \ldots, n_\ell\}$. Fix $j \in \{1, \ldots, k + 1\}$. Introduce edges

$$L(\{I_\ell\}, j) := \{t \in Y \mid t_i(\ell) = z_j \text{ for } \ell \in V, i \in I_\ell\}.$$

Next assume that the graph Δ remains connected after any vertex $\ell \in V$ with $|I_\ell| = 1$ is removed. Under these assumptions, define edges

$$L(\{I_\ell\}) := \{t \in Y \mid t_i(\ell) = t_h(\ell), t_i(\ell) = t_g(m) \text{ for } \ell, m \in V; i, h \in I_\ell; g \in I_m\}.$$

Proposition 5.1 *(1) $L(\{I_\ell\}, j)$, $L(\{I_\ell\})$ are dense.*

(2) Every dense edge has the form above.

Proof. For any graph G with vertices $\{1, \ldots, m\}$ and edges E, associate a central arrangement $\mathcal{A}_G$ in $\mathbb{C}^m$ consisting of $\{x_i = 0 (1 \leq i \leq m)\}$ and $\{x_i = x_j | \{i, j\} \in E\}$. Define a central arrangement $\mathcal{B}_G$ consisting of $\{x_i = x_j | \{i, j\} \in E\}$. (The arrangement $\mathcal{B}_G$ is called a graphic arrangement [7, 2.4].) In order to prove (1) and (2), it is enough to show the following lemma;

Lemma 5.2 *(a) $\mathcal{A}_G$ is not decomposable iff G is connected,*

(b) $\mathcal{B}_G$ is not decomposable iff G is 2–connected, that is, G remains connected after any vertex is removed.

Proof. (a): If G is disconnected, $\mathcal{A}_G$ is obviously decomposable. If G is connected, let T be a maximal tree inside G. Choose an edge $\{i, j\}$ such that j is a terminal point of T. Let $\mathcal{A}'$ and $\mathcal{A}''$ be the deletion and the restriction of $\mathcal{A}_T$ with respect to the hyperplane $\{x_i = x_j\}$. Since $\beta(\mathcal{A}') + \beta(\mathcal{A}'') = \beta(\mathcal{A}_T)$ [2, Theorem1], we can prove $\beta(\mathcal{A}_T) = 1$ for any tree by induction on the number of edges. This shows $\beta(\mathcal{A}_G) \geq \beta(\mathcal{A}_T) = 1$.

(b): Note that the matroid associated with the arrangement $\mathcal{B}_G$ is the same as the matroid associated with the graph G. The matroid is connected if and only if G is 2–connected [9]. $\square$

Let $\mathbb{C}^n = Y - \bigcup_{(i,\ell)\in X} H_{(i,\ell),k+1}$. Let U be the complement in Y to the union of "hyperplanes" of $\mathcal{A}$. Recall $f_i, \omega_i, P_i, \omega, \mathcal{E}$, and $A^\bullet$ from Section 1. ω can be uniquely extended to be an End $\mathbb{C}^s$–coefficient rational 1-form $\overline{\omega}$ on Y. For $(i, \ell) \in X$ the residue of $\overline{\omega}$ at $H_{(i,\ell),k+1}$ is

$$P_{(i,\ell),k+1} = -\sum_{j=1}^{k} P_{(i,\ell),j} - \sum_{\substack{j=1 \\ j\neq i}}^{n_\ell} P_{(j,\ell),(i,\ell)} - \sum P_{(i,\ell),(j,m)}$$

where the last sum is over all m such that v_ℓ and v_m are joined by an edge in Γ and $j = 1, \ldots, n_m$.

For any edge L in $\mathcal{A}$, let P_L be the sum of residues of $\overline{\omega}$ at all "hyperplanes" of $\mathcal{A}$ contaning L.

Theorem 5.3 *Let $\mathcal{L}$ be the set of dense edges of $\mathcal{A}$. Suppose that*

(Mon)* : *for all $L \in \mathcal{L}$, none of the eigenvalues of P_L lies in $\mathbb{N} - \{0\}$.*

Then the inclusion

$$A^\bullet \hookrightarrow \Gamma(U, \Omega^\bullet_U(\mathcal{E}))$$

is quasiisomorphism. $\square$

Corollary 5.4 *Suppose that*

(Mon) :** *for all $L \in \mathcal{L}$, none of the eigenvalues of P_L lies in $\mathbb{N} \cup \{0\}$.*

Also suppose that $P_i P_j = P_j P_i$ for all i, j. Then

$$H^p(U, \mathcal{S}) = 0 \quad \text{for} \quad p \neq n. \quad \square$$

6 Kac-Kazhdan conditions

Let $\mathcal{G}$ be a finite dimensional simple complex Lie algebra with Chevalley generators $e_i, f_i, h_i,\ i = 1, \ldots, r$. Let $\mathcal{G} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$ be the corresponding Cartan decomposition; $\alpha_1, \ldots, \alpha_r \in \mathcal{H}^*$ the simple roots, θ the highest root. Let $(\ ,\ )$ be the symmetric non-degenerate bilinear form on $\mathcal{G}$ such that $(\theta, \theta) = 2$.

Let T be an independent variable, $\mathbb{C}[T]$ the ring of polynomials, $\mathbb{C}[T, T^{-1}]$ the ring of Laurent polynomials. For $f(T), g(T) \in \mathbb{C}[T, T^{-1}]$, set

$$\mathrm{res}_0(f(T)dg(T)) = \text{coefficient at } T^{-1} \text{ in } f(T)g'(T).$$

The space $\mathcal{G} \otimes_{\mathbf{C}} \mathbb{C}[T, T^{-1}]$ is a Lie algebra with bracket

$$[b \otimes f(T), c \otimes g(T)] = [b, c] \otimes f(T)g(T)$$

for $b, c \in \mathcal{G}$. Define $\hat{\mathcal{G}}$ as a central extension of $\mathcal{G} \otimes_{\mathbf{C}} \mathbb{C}[T, T^{-1}]$,

$$\hat{\mathcal{G}} = \mathcal{G} \otimes \mathbb{C}[T, T^{-1}] \oplus \mathbb{C}K,$$

where K is a central element of $\hat{\mathcal{G}}$, and

$$[b \otimes f(T), c \otimes g(T)] = [b, c] \otimes f(T)g(T) + (b, c)\mathrm{res}_0(f(T)dg(T))K.$$

Set $\hat{\mathcal{G}}^+ = \mathcal{G} \otimes \mathbb{C}[T] \oplus \mathbb{C}K$; it is a Lie subalgebra of $\hat{\mathcal{G}}$.

Fix a complex number k. Set $\kappa = k + g$ where g is the dual Coxeter number of $\mathcal{G}$, cf. [5], 6.1.

For $\Lambda \in \mathcal{H}^*$, let $M(\Lambda)$ be the Verma module over $\mathcal{G}$ with highest weight Λ. Consider $M(\Lambda)$ as a $\hat{\mathcal{G}}^+$-module by setting $\mathcal{G} \otimes T\mathbb{C}[T]$ to act as zero and K as multiplication by k. Set

$$\hat{M}(\Lambda) := U(\hat{\mathcal{G}}) \otimes_{U(\hat{\mathcal{G}}^+)} M(\Lambda).$$

It is a Verma module over $\hat{\mathcal{G}}$.

Proposition 6.1 *(Kac-Kazhdan conditions) $\hat{M}(\Lambda)$ is reducible if and only if at least one of the following three conditions is satisfied.*

(1) $\kappa = 0$.

(2) There exist a positive root α of $\mathcal{G}$ and natural numbers $p, s \in \mathbb{N} - \{0\}$ such that

$$(\Lambda, \alpha) + (\rho, \alpha) = p\frac{(\alpha, \alpha)}{2} - (s - 1)\kappa,$$

where ρ is half-sum of positive roots of $\mathcal{G}$.

(3) There exist a positive root α of $\mathcal{G}$ and natural numbers $p, s \in \mathbb{N} - \{0\}$ such that

$$(\Lambda, \alpha) + (\rho, \alpha) = -p\frac{(\alpha, \alpha)}{2} + s\kappa.$$

Proof. We use notations of [5], Ch. 6,7. In these notations the Kac-Kazhdan reducibility condition, [6], Thm 1, reads as

$$\Big\langle \Lambda, \nu^{-1}(\beta)\Big\rangle + \Big\langle \hat\rho, \nu^{-1}(\beta)\Big\rangle - p\frac{(\beta,\beta)}{2} = 0$$

for some positive root β of $\hat{\mathcal{G}}$ and a positive integer p. (Here we denoted by $\hat\rho$ an element denoted by ρ in [5], to distinguish it from our ρ.)

By *loc. cit.*, 6.3, every such β has one of the following forms: (1) $\beta = m\delta,\ m > 0$; (2) $\beta = \alpha + m\delta,\ m \geq 0$; (3) $\beta = -\alpha + m\delta,\ m > 0$, where α is a positive root of $\mathcal{G}$, m an integer. From *loc. cit* it follows easily that $\langle \Lambda, \nu^{-1}(\delta)\rangle = k$, $\langle \hat\rho, \nu^{-1}(\delta)\rangle = g$ and $\langle \hat\rho, \nu^{-1}(\alpha)\rangle = (\rho, \alpha)$. This implies the proposition. $\square$

Let w be the longest element of the Weyl group of $\mathcal{G}$. For $\Lambda \in \mathcal{H}^*$, set $\Lambda' = -w(\Lambda)$.

Proposition 6.2 *$\hat M(\Lambda')$ is reducible if and only if $\hat M(\Lambda)$ is reducible. The Kac-Kazhdan conditions for Λ' expressed in terms of Λ coincide with the Kac-Kazhdan conditions for Λ.*

Proof. For a positive root α, $-w(\alpha)$ is a positive root. This implies the proposition. $\square$

7 Resonances of discriminantal arrangements

Let Γ be the Dynkin diagram of a complex simple Lie algebra $\mathcal{G}$. The vertices of the diagram are labelled by simple roots $\alpha_1, \dots, \alpha_r$ of the algebra. Let $n_1, \dots, n_r$ be nonnegative integers, $n = n_1 + \cdots + n_r$. For pairwise distinct $z_1, \dots, z_k \in \mathbb{C}, z_{k+1} = \infty$, consider in $Y = (\mathbf{P}^1)^n$ the discriminantal arrangement $\mathcal{A}$ associated to these data.

Let $\Lambda_1, \dots, \Lambda_k \in \mathcal{H}^*$. Set $\Lambda_{k+1} = -\omega(\Lambda_1 + \cdots + \Lambda_k - n_1\alpha_1 - \cdots - n_r\alpha_r)$. Fix a nonzero complex number κ. Introduce an integrable connection $d+\omega$ on the trivial bundle $\mathcal{E} := \mathcal{O}_U$ with

$$\omega = \sum_{(i,\ell)\in X}\sum_{j=1}^{k} P_{(i,\ell),j}\omega_{(i,\ell),j} + \sum_{\ell=1}^{r}\sum_{1\leq i<j\leq n_\ell} P_{(i,\ell),(j,\ell)}\omega_{(i,\ell),(j,\ell)} + \sum_{1\leq \ell<m\leq r}\sum_{i=1}^{n_\ell}\sum_{j=1}^{n_m} P_{(i,\ell),(j,m)}\omega_{(i,\ell),(j,m)},$$

where

$$\omega_{(i,\ell),j} = d(t_i(\ell) - z_j)/(t_i(\ell) - z_j), \quad \omega_{(i,\ell),(j,m)} = d(t_i(\ell) - t_j(m))/(t_i(\ell) - t_j(m)),$$

$$P_{(i,\ell),j} = -(\alpha_\ell, \Lambda_j)/\kappa, \quad P_{(i,\ell),(j,m)} = -(\alpha_\ell, \alpha_m)/\kappa,$$

see [8] and [10]. ω extends to be a rational 1-form $\overline\omega$ on Y.

For any edge L in $\mathcal{A}$, let P_L be the sum of residues of $\overline\omega$ at all "hyperplanes" of $\mathcal{A}$ containing L. For $p \in \mathbb{N}\cup\{0\}$, we say that the connection $d+\omega$ has a **resonance at L of level** p, if $P_L = p$.

The following theorem connects resonances of $\mathcal{A}$ with the Kac-Kazhdan conditions for the Verma modules $\hat M(\Lambda_1), \dots, \hat M(\Lambda_{k+1})$ of the affine algebra $\hat{\mathcal{G}}$. Let $\alpha = \sum a_\ell \alpha_\ell$ be a positive root of $\mathcal{G}$, p a natural number. Assume that $a_\ell p \leq n_\ell$ for all ℓ. For every ℓ, fix a subset $I_\ell \subseteq \{1, \dots, n_\ell\}$ consisting of $a_\ell p$ elements.

Theorem 7.1 *(1) For every $j = 1, \dots, k+1$, the edge $L_j = L(\{I_\ell\}, j)$ is dense.*

(2) For $j = 1, \dots, k$ and every natural number s, the resonance condition at L_j of level ps, $P_{L_j} = ps$, coincides with the Kac-Kazhdan condition of type (2) for $\hat{M}(\Lambda_j)$,

$$(\Lambda_j, \alpha) + (\rho, \alpha) = p\frac{(\alpha, \alpha)}{2} - s\kappa.$$

(3) For $j = k+1$ and every natural number s, the resonance condition at L_{k+1} of level ps, $P_{L_{k+1}} = ps$, coincides with the Kac-Kazhdan condition of type (3) for $\hat{M}(\Lambda_{k+1})$,

$$(\Lambda_{k+1}, \alpha) + (\rho, \alpha) = -p\frac{(\alpha, \alpha)}{2} + s\kappa.$$

Remarks. (1) For resonance values of $\Lambda_1, \dots, \Lambda_k, \kappa$, nontrivial cohomological relations occur in the image of $A^\bullet \subset \Gamma(U, \Omega_U(\mathcal{E}))$. The Theorem suggests that the relations correspond to singular vectors in the Verma modules $\hat{M}(\Lambda_1), \dots, \hat{M}(\Lambda_{k+1})$. In [4] this correspondence was established for the simplest singular vector in $\hat{M}(\Lambda_{k+1})$, the correspondence implied algebraic equations satisfied by conformal blocks in the WZW model of conformal field theory.

(2) For $j = 1, \dots, k$ and natural number p, the Kac-Kazhdan condition, $(\Lambda_j, \alpha) + (\rho, \alpha) = p\frac{(\alpha,\alpha)}{2}$, appears as a degeneration condition for a certain contravariant form of the arrangement $\mathcal{A}$, see [8, Secs. 3, 6].

Proof. (1) For a positive root $\alpha = \sum a_\ell \alpha_\ell$ consider the subset $\{\alpha_\ell \mid a_\ell > 0\}$ of the set of simple roots. The subset distinguishes a subgraph of the Dynkin diagram. The subgraph is connected [1, ch. 7, sec. 1]. Now L_j is dense by Proposition 5.1.

(2)

$$\begin{aligned} P_{L_j} - ps &= \frac{1}{\kappa}[(-\Lambda_j, \alpha)p + \sum_{r=1}^{r} \frac{pa_\ell(pa_\ell - 1)}{2}(\alpha_\ell, \alpha_\ell) + \sum_{1 \le \ell < m \le r} pa_\ell pa_m(\alpha_\ell, \alpha_m)] - ps \\ &= \frac{p}{\kappa}\left[-(\Lambda_j, \alpha) + p\frac{(\alpha, \alpha)}{2} - \sum_{\ell=1}^{r} a_\ell \frac{(\alpha_\ell, \alpha_\ell)}{2} - s\kappa\right] \\ &= \frac{p}{\kappa}\left[-(\Lambda_j, \alpha) - (\rho, \alpha) + p\frac{(\alpha_\ell, \alpha_\ell)}{2} - s\kappa\right]. \end{aligned}$$

This proves (2). Part (3) is proved by similar direct computations using Proposition 6.2. □

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